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AXIOMATIC DEFINITION OF SETS

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Abstract. The aim of this paper is to give an alternative definition of sets as follows: A domain is a *set* if and only if it belongs to *Set*.

1. Introduction

According to [3, 6], set theory was invented by Georg Cantor in his main publications appearing between 1874 and 1897. Between 1895 and 1910 a number of contradictions were discovered in various parts of set theory by B. Russell, C. Burali-Forti, G. G. Berry and G. Cantor himself. The discovery of the antinomy made it clear that a revision of the principles of Cantor set theory was necessary. The attempt to improve set theory which is best known among mathematicians is the axiomatic theory first set forth by E. Zermelo in 1908. After Zermelo, the axiomatic set theory has been developed by T. Skolem, A. Fraenkel, J. von Neumann, P. Beranys, K Gödel, etc.

Zermelo-Fraenkel set theory, with certain modification due to T. Skolem and A. Fraenkel and J. von Neumann, is widely used up to the present day (*cf.* [2, 4, 5] and see also [8]).

E. Zermelo did not decisively give the definition of a set but constructed sets using axioms. On the other hand, an axiomatic definition of a set was given by von Neumann as follows(*cf.* [3, 4, 6]):

A class is a set if and only if it belongs to a class.

Nevertheless, mathematicians are still forced to face two questions:

(1) What properties legitimately define sets?

(2) Does there exist the class (or set) X of sets which does not have the peculiar property that $X \in X$?

The aim of this paper is to give an alternative answer to these questions as follows:

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A domain is a set if and only if it belongs to Set.

Further we show that Set provides a firm foundation for a system of set theory which include all of Cantor's basic results as well as the constructions needed for contemporary mathematics.

This paper is organized as follows: Section 2 is devoted to give notations, axioms and definitions. In section 3, we give two concepts: one is the concept of domain, and the other is a concept of sieve. We adopt axiom of sieve to guarantee the existence of sieve and then we show that every sieve implies all axioms of Zermelo-Fraenkel set theory except axioms of regularity and replacement. In section 4, we give an alternative definition of ordinals and show that every sieve holds for significant consequences of axioms of replacement. Section 5 is devoted to define a concept of sets.

2. Preliminaries

Terminology and theorems are adopted from [2, 4, 5, 8] if not explained in this paper. In this section, we give basic and important notations, definitions and two axioms and choose two undefined notions: the word *class* and *membership relation* $\in$, which is read 'is an element of' or 'belongs to.' From here on, lower-case letters s, t, x, y, ... will be used only to designate elements and capital-letters X, Y, ... may denote either an element or a class which is not an element.

Definition 2.1. $x \notin X$ if and only if it is false that $x \in X$.

Definition 2.2. $X = Y$ if and only if for each z, $z \in X$ if and only if $z \in Y$

Definition 2.3. $X \neq Y$ if it is false that $X = Y$.

Definition 2.4. $X \subseteq Y$ if and only if for each z, if $z \in X$, then $z \in Y$. In this case, X is called a *subclass* of Y.

Definition 2.5. $X \subset Y$ if and only if $X \subseteq Y$ and $X \neq Y$. In this case, X is called a *proper* subclass of Y.

In the rest of this section, we state two axioms which are founded in [4] (see also [7]) and give a few of their elementary consequences.

Axiom of Extensionality. *If $x = y$ and $x \in Z$, then $y \in Z$.*

Axiom of Classification. *Let $P(x)$ be built up from atomic propositions of the form $s \in t$ by use of the logical connectives $\vee$, $\wedge$, $\neg$, $\rightarrow$ (or, and, not, if-then), the quantifiers $\exists$, $\forall$ (for some, for all), brackets and variables x, y, ..., A, B, ... Then for each x,*

$$x \in \{y : P(y)\} \text{ if and only if } P(x) \text{ and } x \in z \text{ for some class } z.$$

Throughout this paper, we need the following definitions:

Definition 2.6.

1) $\emptyset$ is a unique class such that for each x, $x \notin \emptyset$.
2) For any class X, $p(X)$ is a unique class such that $z \in p(X)$ if and only if $z \subseteq X$.
3) For any class X, $\cup X$ is a unique class such that $s \in \cup X$ if and only if there exists $x \in X$ such that $s \in x$.
4) For any classes s and t, X is a unique class such that $x \in X$ if and only if $x = s$ or $x = t$.
5) For any classes X and Y, $X - Y$ is a unique class such that $x \in X - Y$ if and only if $x \in X$ and $x \notin Y$.
6) For any classes X and Y, $X \times Y$ is a unique class such that $z \in X \times Y$ if and only if $z = (x, y)$, $x \in X$ and $y \in Y$, where (x, y) denotes $\{\{x\}, \{x, y\}\}$.

Definition 2.7. A subclass $f \subseteq X \times Y$ is called a *function* from X to Y if it satisfies the following conditions:

1) for each $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$.
2) if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

In this case, we write $f : X \rightarrow Y$, and $y = f(x)$ stands for $(x, y) \in f$. In particular, if $X = Y$, then f is called an *unary operation* on X, and $x^f = y$ stands for $(x, y) \in f$.

3. Domains and sieves

In this section, we give two concepts: One is a concept of domain, and the other is a concept of sieve. We show that every sieve implies all axioms of Zermelo-Fraenkel set theory except both axiom of regularity and axiom of replacement, and every sieve holds significant consequences of axiom of regularity. For these purpose, we first introduce a concept of domain which implies the most significant consequence of regularity axiom.

Definition 3.1. A class n is called a *chain* if it satisfies the following:

1) there exists an element $x \in n$ such that $x \in x$ and $x \in s$ for all $s \in n$,
2) if $s \in n$ and $t \in n$, then $s \in t$ or $t \in s$, and
3) there exists an element $e \in n$ such that $t \in e$ for all $t \in n$.
4) if $s \in n$ then there is $t \in n$ such that $z = s$ or $z = t$ whenever $s \in z \in t$.

NOTATION. For a chain n, the class $e \in n$ satisfying condition 3) of the above definition denotes e_n. That is, $e = e_n$.

EXAMPLE 3.2.

1) *If $x \in x$, then $\{x\}$ is a chain and $e_{\{x\}} = x$.*
2) *Let n be the class such that $t \in n$ if and only if $t = x$ or $t = y$. If $x \in x$ and $x \in y$, then n is a chain and $e_n = y$.*

DEFINITION 3.3. A class X is called *proper* if there is a chain n such that $e_n \in X$. Otherwise, it is called a *domain*.

EXAMPLE 3.4.

1) *Every chain is proper.*
2) *$\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are domains.*

REMARK 3.5. If X is a domain, there is no class s such that $s \in s \in X$. and so every domain does not belong to itself. This is a significant consequence of axiom of regularity.

Now we characterize properties of domains:

THEOREM 3.6.

1) *A class X is a domain if and only if every element of X is a domain.*
2) *A class X is a domain if and only if $p(X)$ is a domain.*

Proof. 1) Let us assume that there exists a proper class t of X. Then there is a chain n such that $e_n \in t$. Let $m = n \cup \{t\}$. Then m is a chain such that $e_m = t$. However, it is impossible because X is a domain and $t = e_m \in X$. The other implication is immediate from the definition of domains.

2) Suppose $p(X)$ is proper, then there is a chain n such that $e_n \in p(X)$. If $n = \{e_n\}$, then $e_n \in e_n$ and hence $e_n \in X$ because $e_n \subseteq X$. Thus X is proper, which is impossible because X is a domain. If $\{e_n\} \subset n$, then since n is a chain, there is an element $t \in n$ such that $z = t$ and $z \neq e_n$ whenever $t \in z \in e_n$. Let Let $m = n - \{e_n\}$. Then m is a chain such that $e_m = t$. Since $e_n \subseteq X$ and $e_m \in e_n$, $e_m \in X$ and hence X is proper, which is impossible, because X is a domain. Thus $p(X)$ is a domain. The converse is immediate from part 1) of this theorem. $\square$

The following is immediate from the above theorem:

Corollary 3.7. *One has the following:*

1) *X and Y are domains if and only if $\{X, Y\}$ is a domain.*
2) *X and Y are domains if and only if $X \cup Y$ is a domain.*
3) *X is domain if and only if $\cup X$ is a domain.*
4) *If Z is a class of domains, then there is a domain Y such that $Y \notin Z$.*

Remark 3.8.

1) By Example 3.4.2, Theorem 3.6 and Corollary 3.7, there exist infinitely many domains.
2) By part 4) of the above corollary, the class of all the domains does not exist.

The following definitions are the most essential concepts in this paper.

Definition 3.9. Let S and X be domains. Then X is said to be:

1) *S-weak transitive* if there exists $z \in S$ such that $t \subsetneq z$ whenever $t \in X$.
2) *S-transitive* if $X \subset S$ and $x \subset X$ whenever $x \in X$.

Remark 3.10.

1) It is easy to show that if $T \subseteq S$, then every T-(weak) transitive domain is S-(weak) transitive.
2) It is easy to show that if x is S-transitive, then $\cup x \subseteq x$.
3) If $x \in S$ and x is S-transitive, then x is S-weak transitive.

Notation. For a S-transitive domain X, let

$$l(X) = \cup\{t \in X : t = \cup t\}.$$

Using Definition 3.9 and the above notation, we define the main concept as follows:

Definition 3.11. A domain S is called a *sieve* if it satisfies the following conditions:

1) $S \neq \emptyset$.
2) For any domain x and y, not necessarily distinct, $x \in S$ and $y \in S$ if and only if (T_1) $x \cup y$ is S-weak transitive or (T_2) S-transitive such that $l(x) \in x$ and $l(y) \in y$.

In order to guarantee the existence of the sieve, we now adopt the following axiom:

Axiom of Sieve. *There exists a sieve.*

In the rest of this section, S denotes a sieve and we assume that every class is a domain.

Remark 3.12.

1) Note that if $s = t$, then $s \cup t = t$. Hence, by condition 2) of Definition 3.11, it is clear that $t \in S$ if and only if t is S-weak transitive or S-transitive such that $l(t) \in t$.
2) It is clear that every element of S is S-weak transitive.
3) Since S is a domain, $S \notin S$ and, by Theorem 3.6, every element of S is also a domain, so if $x \in S$, then $x \notin x$.

Now, using the above remark, we characterize the properties of S:

Theorem 3.13. *One has the following:*

1) $\emptyset \in S$.
2) *If* $x \in S$ *and* $y \subset x$*, then* $y \in S$.
3) $x \subset S$ *whenever* $x \in S$.
4) $x \in S$ *if and only if* $p(x) \in S$.
5) $x \in S$ *if and only if* $\cup x \in S$.
6) $x \in S$ *and* $y \in S$ *if and only if* $\{x, y\} \in S$.

Proof. 1) Since $S \neq \emptyset$, $\emptyset$ is S-weak transitive and hence $\emptyset \in S$.

2) If x is S-weak transitive, then there exists $z \in S$ such that $x \subseteq p(z)$ and so $y \subseteq p(z)$ because $y \subseteq x$. Thus y is also S-weak transitive and hence $y \in S$. If x is S-transitive, then $s \subset x$ for all $s \in x$. Since $y \subseteq x$, $t \subset x$ for all $t \in y$. Since $x \in S$, y is S-weak transitive and so $y \in S$.

3) If x is S-transitive, then $s \subset x$ for all $s \in x$ and hence, by part 2) of this theorem, $s \in S$ for all $s \in x$. Since S is domain, $x \subset S$. If x is S-weak transitive, there exists $z \in S$ such that $x \subseteq p(z)$. Since $z \in S$ and S is a domain, $x \subset S$.

4) Since $x \in S$, $p(x)$ is S-weak transitive and so $p(x) \in S$. The converse is immediate from part 3) of this theorem.

5) If x is S-weak transitive, then there exists $z \in S$ such that $x \subseteq p(z)$. Since, for each $s \in \cup x$, there exists $a \in x$ such that $s \in a$, $s \in z$ and hence $\cup x \subseteq z$. Since $z \in S$, by part 2) of this theorem, $\cup x \in S$. If x is S-transitive, then $\cup x \subseteq x$. Since $x \in S$, by part 2) of this theorem, $\cup x \in S$. The converse is immediate from the definition of $\cup x$ and part 1) of Remark 3.12.

6) If $x \in S$ and $y \in S$, then, by part 1) of Remark 3.12, $x \cup y \in S$ and hence, by part 4) of this theorem, $p(x \cup y) \in S$. Since $\{x, y\} \subseteq p(x \cup y)$,

by part 2) of this theorem, $\{x, y\} \in S$. The converse is immediate from part 3) of this theorem. $\square$

REMARK 3.14.

1) It is immediate from part 6) of the above theorem that $\{x\} \in S$ whenever $x \in S$.
2) Part 3) of the above theorem means that every element of S is hereditarily in S (*cf.* [1]).

4. S-ordinals

In this section, we give an alternative definition of ordinals and every sieve holds significant consequences of axiom of replacement. We first modify a definition of ordinals as follows (*cf.* [1], [2], [4] and [8]):

DEFINITION 4.1.

1) A domain X is a *S-ordinal* if it satisfies the following:
   O_1) it is S-transitive,
   O_2) its elements are S-transitive, and
   O_3) $tr(X) \subseteq X$, where $tr(X)$ denotes the domain all of whose elements are S-transitive proper subclass of X.
2) An S-ordinal X is called a *limit S-ordinal* if $X = \cup X$. Otherwise, it is called a *successor S-ordinal*.

The proof of the following is exactly the same as the proof of Theorem 110 in [4].

THEOREM 4.2. *If x is a S-ordinal, y is a S-ordinal and $x \neq y$ then $x \in y$ or $y \in x$.*

By axiom of classification and the above theorem, there exists the class Or_S of all S-ordinals. It is clear that $Or_S = \cup Or_S$. Also, since $\emptyset$ is a S-ordinal, $\{\{\emptyset\}\} \in S$ and $\{\{\emptyset\}\} \notin Or_S$, $\emptyset \neq Or_S \subset S$. Thus Or_S is a S-ordinal and $Or_S \notin Or_S$. We now show that Or_S is the only S-ordinal which does not belong to S.

THEOREM 4.3. $Or_S \notin S$.

Proof. Suppose $Or_S \in S$, then $Or_S \in Or_S$. This is impossible. Hence $Or_S \notin S$.

$\square$

REMARK 4.4. The above theorem means that Or_S is not S-weak transitive and $l(Or_S) = Or_S$. Theorems 4.2 and 4.3 mean that Or_S is the only S-ordinal which does not belong to S.

Now we consider the axiom of replacement. It is well known [3, 6] that A. Fraenkel and T. Skolem had independently proposed adjoining replacement axiom to establish that

$$\mathbf{E}_\omega = \{\omega, p(\omega), p(p(\omega)), ...\}$$

be a set since, as they pointed out, Zermelo's axioms cannot establish this. However, even $\mathbf{E}_\emptyset$ cannot be proved to be a set from Zermelo's axioms. Also the ordinal number $\omega 2$, which is the set of all $\omega + n$ for all $n \in \omega$, is the first ordinal that cannot be constructed without Replacement. In fact, Replacement has been latterly regarded as somehow less necessary or crucial than the other axioms, the purported effect of the axiom being only on large-cardinality sets [6].

In the rest of this section, we show that ordinals belong to S, including ω and $\omega 2$, and $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, ...\}$ belongs to S, and if δ is a limit ordinal which belongs to S, then

$$\mathbf{E}_\delta = \{\delta, p(\delta), p(p(\delta)), ...\}$$

belongs to S.

We begin by giving a basic definition to define inductive domains:

Definition 4.5. Let X be a domain and f an unary operation on X. Then a S-transitive element a of X is called the *initial element* with respect to f if $x^f \neq a$ for each $x \in X$.

Notation. For a domain X and f an unary operation on X, i_f^X denotes the domain of initial elements of X with respect to f.

Definition 4.6. The triple (X, f, a) is called an *inductive domain*, where X is a domain, f is an unary operation on X such that $x^f \in X$ whenever $x \in X$ and $a \in i_f^X$.

In the next theorem, we show that there exists a domain which implies the principle of mathematical induction stated as an axiom of the natural numbers.

Theorem 4.7. *Let (X, f, a) be an inductive domain. Then there exists the inductive domain (G, f, a) such that if $Y \subseteq G \subseteq X$ and (Y, f, a) is an inductive domain, then $G = Y$.*

Proof. Let G be the domain such that $y \in G$ if and only if $y \in Z$ whenever (Z, f, a) is an inductive domain. Since (X, f, a) is an inductive domain, $G \subseteq X$ and hence $i_f^X \subseteq i_f^G$. Thus $a \in i_f^G$. It is clear that $g^f \in G$ whenever $g \in G$. Thus (G, f, a) is an inductive domain. If $Y \subseteq G$ and (Y, f, a) is an inductive domain, then, by the definition of G, $G = Y$. $\square$

Definition 4.8. The inductive domain (G, f, a) given in the proof of the above theorem is called a *Peano domain* of (X, f, a).

Remark 4.9. It is clear that (G, f, a) is a Peano domain then $G \subseteq \cup G$.

In a sense, the concept of Peano domain is a generalization of Vaugh's Peano structure (*cf.* [8]).

Theorem 4.10. *Let $+$ be an unary operation on S defined by $a^+ = a \cup \{a\}$ for all $a \in S$. Then one has the following:*

1) *$(S, +, \emptyset)$ is an inductive domain.*
2) *$\cup S = S$.*
3) *There exists a Peano domain $(\omega, +, \emptyset)$ such that $\omega \subset S$.*

Proof. 1) Clearly $\emptyset \in i_+^S$. Suppose $a \in S$, then by part 1) of Remark 3.14 and condition 2) of Definition 3.11, $a^+ = a \cup \{a\} \in S$. Thus $(S, +, \emptyset)$ is an inductive domain.

2) It is immediate from 1) of this theorem and part 3) of Theorem 3.13.

3) Let $T = S - \{\{\{\emptyset\}\}\}$. Then $(T, +, \emptyset)$ is an inductive domain of $(S, +, \emptyset)$, because $a^+ \neq \{\{\emptyset\}\}$ for all $a \in S$. Hence by Theorem 4.7, there exists the Peano domain $(\omega, +, \emptyset)$ such that $\omega \subset S$. □

Theorem 4.11. *Let b be an unary operation on S defined by $x^b = \{x\}$ for all $x \in S$. Then one has the following:*

1) *$(S, b, \emptyset)$ is an inductive domain.*
2) *There exists a Peano domain $(\beta, b, \emptyset)$ such that $\beta \subset S$.*

Proof. 1) Clearly $\emptyset \in i_b^S$. Suppose $a \in S$, then by part 1) of Remark 3.14, $a^b = \{a\} \in S$. Thus $(S, b, \emptyset)$ is inductive domain.

2) Let $T = S - \{\{\emptyset, \{\emptyset\}\}\}$. Then $(T, b, \emptyset)$ is an inductive domain of $(S, b, \emptyset)$, because $a^b \neq \{\emptyset, \{\emptyset\}\}$ for all $a \in S$. Hence by Theorem 4.7, there exists the Peano domain $(\beta, b, \emptyset)$ such that $\beta \subset S$. □

Remark 4.12. 1) ω is just the set of all the natural numbers.
2) $\beta = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, ...\}$.

Theorem 4.13. *Let $(X, f, \emptyset)$ be a Peano domain such that $t \in x^f$ if and only if $t \in x$ or $t = x$. Then if $X \subset S$ and $l(X) \in X$, then $X \in S$.*

Proof. Let $Z = \{x \in X : x \subset X\}$. It is clear that $\emptyset \in Z$. Suppose $x \in Z$ and $t \in x^f$, then $t = x$ or $t \in x$, and hence $t \in X$. Thus $x^f \subset X$. Since $x^f \in X$, $x^f \in Z$. Hence $Z = X$. Since $X \subset S$, X is S-transitive. Since $l(X) \in X$. Thus $X \in S$. □

Using Theorems 4.10, 4.11 and the above theorem, we have at once the following:

Corollary 4.14.

1) *ω is a limit S-ordinal in S.*

2) *$\beta \in S$.*

Proof. 1) By part 3) of Theorem 4.10, $\omega \subset S$ and it is clear that $t \in x^+$ if and only if $t \in x$ or $t = x$. Since $l(\omega) = \emptyset \in \omega$. Thus, by the above theorem, $\omega \in S$.

2) By part 2) of Theorem 4.11, $\beta \subset S$ and it is clear that $t \in x^b$ if and only if $t = x$. Since $l(\beta) = \{\emptyset\} \in \beta$. Thus, by the above theorem, $\beta \in S$. □

Lemma 4.15. *Let δ be a limit S-ordinal in S and $(\delta 2, +, \delta)$ a Peano domain. Then $\delta 2 \in S$.*

Proof. Since $\delta 2 \subset S$ and $l(\delta 2) = \delta \in \delta 2$, $\delta 2 \in S$. □

Just as for Zermelo's sets, we can now introduce the recursion theorem for domain as follows (*cf.* [1], [2], [4] and [8]):

Theorem 4.16 (Recursion Theorem for Domain). *Let A be a domain, z a fixed element of A, and f a function from A to A. Then there exists a unique function $\gamma : \omega \to A$ such that*

1. $\gamma(\emptyset) = z$, and

2. $\gamma(n^+) = f(\gamma(n))$, $n \in \omega$.

The following is immediate from the above theorem:

Corollary 4.17. *1) For the Peano domain (E_a, p, a), where $x^p = p(x)$, there exists a unique bijection $\delta_a : \omega \to \mathbf{E}_a$ defined by the two conditions:*

1. $\delta_a(\emptyset) = a$, and

2. $\delta_a(n^+) = [(\delta_a(n)]^p$, $n \in \omega$.

Notation. $\delta_a(n) = p^n(a)$.

Lemma 4.18. *Let a be a domain. Then for each $n \in \omega$, $a + n^+ \notin p(p^n(a))$.*

Proof. Since a is a domain, $a \notin a$. Let $N = \{n \in \omega : a + n^+ \notin p(p^n(a))\}$. Suppose $a^+ \in p(a)$, then $a^+ \subseteq a$. Since $a \in a^+$, $a \in a$. This is impossible because $a \notin a$. Thus $\emptyset \in N$. Suppose $n \in N$ and $a + [n^+]^+ \in p(p^{n^+}(a))$, then

$$a + [n^+]^+ \subseteq p(p^n(a))$$

and so $a+n^+ \in p(p^n(a))$. This is impossible, because $a+n^+ \notin p(p^n(a))$. Thus $N = \omega$. $\square$

THEOREM 4.19. *Let (X, f, a) be a Peano domain such that $l(\cup X) \in \cup X$ and $\cup X \subset S$, $t \in x^f$ if and only if $t \subseteq x$. Then $X \in S$.*

Proof. Let Y be the class such that $y \in Y$ if and only if $y \in X$ and y is S-transitive. It is clear that $a \in Y$. Suppose $x \in Y$ and take any $t \in x^f$, then $t \subseteq x$ and hence if $s \in t$, then $s \in x$. Since x is S-transitive, $s \subset x$ and so $s \in x^f$. Thus $t \subset x^f$ because x^f is a domain. Therefore $X = Y$ and hence $\cup X$ is also S-transitive. Since $l(\cup X) \in \cup X$ and $\cup X \subset S$, $\cup X \in S$. By part 2) of Theorem 3.13 and Remark 4.9, $X \in S$. $\square$

COROLLARY 4.20. *Let δ be a limit S-ordinal and (E_δ, p, δ) a Peano domain. Then $E_\delta \in S$.*

Proof. It is clear that for each $x \in E_\delta$, $t \in x^p$ if and only if $t \subseteq x$ and $l(\cup E_\delta) = \delta \in E_\delta$. Suppose $\delta 2 \in \cup E_\delta$, then $\delta 2 \in p(p^n(\delta))$ for some $n \in \omega$. Then $\delta + n^+ \in p(p^n(\delta))$ since $\delta + n^+ \in \delta 2$. This is impossible because of Lemma 4.18. Thus $\delta 2 \notin \cup E_\delta$. Thus by the above theorem, $E_\delta \in S$. $\square$

REMARK 4.21. It immediately follows from the above corollary that $\cup E_\emptyset \in S$, $E_\emptyset \in S$, $\cup E_\omega \in S$ and $E_\omega \in S$.

5. Definition of sets

In this section, we show that there exists the smallest sieve with respect to $\subseteq$. Using the smallest sieve, we give a definition of sets.

THEOREM 5.1. *There is the sieve S such that, for each sieve D, $S \subseteq D$.*

Proof. Let S be a class such that $x \in S$ if and only if $x \in D$ for every sieve D. Then it is a clear that for each sieve D, $S \subseteq D$ and D is a domain. Suppose $x \in S$ and $y \in S$, then $x \in D$ and $y \in D$ for every sieve D. Since D is a sieve, $x \cup y \in D$ and hence $x \cup y$ is D-weak transitive. That is, for each sieve D, there is an element $z_D \in D$ such that $t \subseteq z_D$ whenever $t \in x \cup y$. Let z be a class such that $t \in z$ if and only if $t \in z_D$ for every z_D. It is clear that, for each z_D, $z \subseteq z_D$ and so $z \in D$ for every sieve D. Hence $z \in S$ and $t \subseteq z$ whenever $t \in x \cup y$. That is, $x \cup y$ is S-weak transitive. Conversely, suppose $x \cup y$ is S-weak transitive, then $x \cup y$ is D-weak transitive for every sieve D, and so $x \in D$ and $y \in D$ for every sieve D. Thus, by the definition of S, $x \in S$ and $y \in S$. Suppose

$x \cup y$ is S-transitive, since, for each sieve D, $S \subseteq D$, $x \cup y$ is D-transitive for every sieve D and so $x \in D$ and $y \in D$ for every sieve D. Thus, by the definition of S, $x \in S$ and $y \in S$. In all, S is a sieve and for each sieve D, $S \subseteq D$. □

Notation. The sieve given in the above theorem is denoted by Set.

Using Theorem 5.1 and the above notation, we can give the main result of this paper as follows:

Definition 5.2. A domain x is called a *set* if $x \in Set$.

It is clear that Set is neither S-weak transitive nor S-transitive and hence Set is a not set.

Finally we adopt axiom of choice:

Axiom of Choice. *For any set x, there exists a function f defined on x such that $f(t) \in t$ for all $t \in x$ such that $t \neq \emptyset$.*

Notation. Set_C is Set plus axiom of choice.

Conclusion Remark. According to Definition 3.11, every set is completely determined by two properties of transitivity and the concept of domain. But the axiom of choice is not necessary to define set itself. Condition (T_1) of Definition 3.11 implies axioms: subset, union and power and condition (T_2) of Definition 3.11 implies axiom of infinity and the significant consequences of the axiom of replacement. Moreover, Conditions (T_1) and (T_2) implies that every S-ordinal except Or_S is a set. The concept of domain implies that every set x satisfies the property $x \notin x$ which is the important consequence of axiom of regularity. Condition 2) of Definition 3.11 implies axiom of paring. By axiom of classification, $\{x \in S : P(x)\}$ exists. Consequently, we conclude that, using only the laws of logic, Set_C provides a firm foundation for a system of set theory which include all of Cantor's basic results as well as the constructions needed for contemporary mathematics.

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